Ten of the 13 numbered statements below if put in the right order prove this theorem:

**Theorem 1** *Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers such that*

$$a_n \to a \text{ as } n \to \infty \quad b_n \to b \text{ as } n \to \infty$$

*Then*

$$a_n - b_n \to a - b \text{ as } n \to \infty$$

## A correct sequence

1. Suppose $\epsilon > 0$
2. We must find $N$ such that if $n > N$ then $|(a_n - b_n) - (a - b)| < \epsilon$
3. Because $a_n \to a$ as $n \to \infty$ there exists $N_1 > 0$ such that if $n > N_1$ then $|a_n - a| < \epsilon/2$
4. Because $b_n \to b$ as $n \to \infty$ there exists $N_2 > 0$ such that if $n > N_2$ then $|b_n - b| < \epsilon/2$
5. Put $N = \max(N_1, N_2)$
6. If $n > N$ then

$$\begin{aligned} |(a_n - b_n) - (a - b)| &= |(a_n - a) - (b_n - b)| && (7) \\ &\leq |a_n - a| + |b_n - b| && (8) \\ &< \epsilon/2 + \epsilon/2 && (9) \\ &= \epsilon \qquad \text{as required} && (10) \end{aligned}$$

Items 2, 3 and 4 can come in any order but must follow 1 and precede 5. Otherwise the order is fixed.

## Spoilers

11. Because $a_n \to a$ there exists $N$ such that if $n > N$ then $|a_n - a| < \epsilon$
12. $\leq |a_n - a| - |b_n - b|$
13. Put $N = N_1 - N_2$

## Feedback

**Using Statement 11:** This is true but does not help satisfy Statement 2, which expresses the destination a rigorous proof must arrive at.

**Using Statement 12:** Think more carefully about modulus signs. It cannot be true in general that $|x - y|$ is smaller than $|x| - |y|$ as $|x - y|$ is non-negative whereas $|x| - |y|$ could be negative — if $x = 1$ and $y = 2$ for example.

**Using Statement 13:** The bigger the value of $n$, the more likely it is that the inequality in Statement 2 is satisfied so $N$ given by $N_1 - N_2$ is likely to allow in too small values of $n$ in Statement 6. Statement 5 could be replaced by "Put $N = N_1 + N_2$" as this would ensure values of $n$ allowed by Statement 6 are big enough.